LECTURES ABOUT (ADVANCED) STATISTICAL PHYSICS

T.S.Biró, *MTA Wigner Research Centre for Physics, Budapest*

Lectures given at: University of Johannesburg, South-Africa,

November 26 – November 29, 2012.

1. **Ancient Thermodynamics (… - 1870)**
2. **The Rise of Statistical Physics (1890 – 1920)**
3. **Modern (postwar) Problems (1940 – 1980)**
4. **Corrections (1950 – 2005)**
5. **Generalizations (1960 – 2010)**
6. **High Energy Physics (1950 – 2010)**

LECTURE THREE ABOUT (ADVANCED) STATISTICAL PHYSICS

T.S.Biró, *MTA Wigner Research Centre for Physics, Budapest*

Lectures given at: University of Johannesburg, South-Africa,

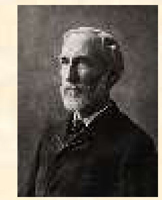
November 28, 2012.

GENERALIZATIONS

- Composition Rules
- Associative Limit
- Zeroth-Law Compatibility
- Universal Thermostat Independence

Entropy formulas

- $S = ln \frac{N!}{\prod_i N_i!}$ Boltzmann (permutation)

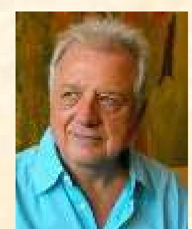


- $S = - \sum P_i \, ln \, P_i$ Gibbs (Planck)

- $S = \frac{1}{1-q} \ln \sum P_i^q$ Rényi

- $S = \frac{1}{q-1} \sum (P_i - P_i^q)$ Tsallis (Chravda, Aczél, Daróczy,…)

There are (much) more !



Canonical distribution with Rényi entropy

$$\frac{1}{1-q}\ln\sum p_i^q - \alpha\sum p_i - \beta\sum p_i E_i = \max$$

$$\frac{1}{1-q}\frac{q\,p_i^{q-1}}{\sum p_i^q} = \alpha + \beta E_i$$

This cut power-law distribution is

an **excellent** fit to particle spectra

in high-energy experiments!

$$p_i = \frac{1}{e^{\hat{L}(S)}}\left(1+(1-q)\frac{\beta(E_i - \langle E\rangle)}{q}\right)^{\frac{1}{q-1}}$$

Canonical distribution with Tsallis entropy

$$\frac{1}{1-q}\sum(p_i^q - p_i) - \alpha\sum p_i - \beta\sum p_i E_i = \max$$

$$\frac{1}{1-q}\, q\, p_i^{q-1} = \alpha + \frac{1}{1-q} + \beta E_i$$

$$p_i = \left(Z^{1-q} + (1-q)\frac{\beta E_i}{q} \right)^{\frac{1}{q-1}}$$

This cut power-law distribution is an excellent fit to particle spectra in high-energy experiments!

Why to use the Tsallis / Rényi entropy formulas? ☺

- It generalizes the Boltzmann-Gibbs-Shannon formula
- It treats statistical entanglement between subsystem and reservoir (due to conservation)
- It claims to be universal (applicable for whatever material quality of the reservoir)
- It leads to a cut power-law energy distribution in the canonical treatment

Why not to use the Tsallis / Rényi entropy formulas? ☹

- They lack 300 years of classical thermodynamic foundation
- Tsallis is not additive, Rényi is not linear
- There is an extra parameter q (mysterious?)
- How do different q systems equilibrate ?
- Why this and not any other ?
- It looks pretty much formal…

Again the Zeroth Law: θ(E1,...)=θ(E2,...)

$$dS_{12} = \frac{\partial S_{12}}{\partial E_1} dE_1 + \frac{\partial S_{12}}{\partial E_2} dE_2 + \ldots = 0$$

$$dE_{12} = \frac{\partial E_{12}}{\partial E_1} dE_1 + \frac{\partial E_{12}}{\partial E_2} dE_2 = 0$$

$$\frac{\partial E_{12}}{\partial E_2} \frac{\partial S_{12}}{\partial S_1} S_1' = \frac{\partial E_{12}}{\partial E_1} \frac{\partial S_{12}}{\partial S_2} S_2'$$

Factorization = ?

The temperature for non-additive composition rules

$$\frac{\partial S_{12}}{\partial S_1}\frac{\partial E_{12}}{\partial E_2}S_1'(E_1) = \frac{\partial S_{12}}{\partial S_2}\frac{\partial E_{12}}{\partial E_1}S_2'(E_2)$$

$$F_1 G_2 H_{12} \cdot A_2 B_1 C_{21} \cdot S_1' = F_2 G_1 H_{21} \cdot A_1 B_2 C_{12} \cdot S_2'$$

$$\frac{H_{12}(S_1,S_2)}{H_{21}(S_1,S_2)} = \frac{C_{12}(E_1,E_2)}{C_{21}(E_1,E_2)} = \text{const.}$$

The temperature for non-additive composition rules

$$\frac{H_{12}(S_1,S_2)}{H_{21}(S_1,S_2)} = \frac{C_{12}(E_1,E_2)}{C_{21}(E_1,E_2)} = \text{const.} = 1$$

$$\frac{F_1(S_1)}{G_1(S_1)} \cdot \frac{B_1(E_1)}{A_1(E_1)} S_1'(E_1) = \frac{F_2(S_2)}{G_2(S_2)} \cdot \frac{B_2(E_2)}{A_2(E_2)} S_2'(E_2)$$

$$\frac{1}{T_1} = \frac{\partial \hat{L}_1(S_1)}{\partial L_1(E_1)} = \frac{\partial \hat{L}_2(S_2)}{\partial L_2(E_2)} = \frac{1}{T_2}$$

Generalized absolute temperature

$$\frac{1}{T} = \frac{\partial \hat{L}(S)}{\partial L(E)}$$

$$\hat{L}(S) = \int \frac{F(S)}{G(S)} dS$$

$$L(E) = \int \frac{A(E)}{B(E)} dE$$

Admissible composition rules

$$H_{12} = \frac{1}{G_2 F_1} \frac{\partial}{\partial S_1} S_{12} = \frac{1}{G_1 F_2} \frac{\partial}{\partial S_2} S_{12} = H_{21}$$

$$\frac{G_1}{F_1} \frac{\partial}{\partial S_1} S_{12} = \frac{G_2}{F_2} \frac{\partial}{\partial S_2} S_{12}$$

$$\frac{\partial}{\partial \hat{L}_1} S_{12} = \frac{\partial}{\partial \hat{L}_2} S_{12}$$

$$S_{12} = \Psi(\hat{L}_1 + \hat{L}_2)$$

$$\hat{L}_{12}(S_{12}) = \hat{L}_1(S_1) + \hat{L}_2(S_2)$$

Admissible composition rules

$$C_{12} = \frac{1}{B_2 A_1} \frac{\partial}{\partial S_1} E_{12} = \frac{1}{B_1 A_2} \frac{\partial}{\partial E_2} E_{12} = C_{21}$$

$$\frac{B_1}{A_1} \frac{\partial}{\partial E_1} E_{12} = \frac{B_2}{A_2} \frac{\partial}{\partial E_2} E_{12}$$

$$\frac{\partial}{\partial L_1} E_{12} = \frac{\partial}{\partial L_2} E_{12}$$

$$E_{12} = \Phi(L_1 + L_2)$$

$$L_{12}(E_{12}) = L_1(E_1) + L_2(E_2)$$

Example: Tsallis entropy

$$S_{12} = S_1 + S_2 + \hat{a} S_1 S_2$$

$$(1 + \hat{a} S_{12}) = (1 + \hat{a} S_1) \cdot (1 + \hat{a} S_2)$$

$$\hat{L}(S) = \frac{1}{\hat{a}} \ln\left(1 + \hat{a} S\right)$$

Heterogeneous equilibrium

$$\frac{1}{\hat{a}_{12}}\ln\left(1+\hat{a}_{12}S_{12}\right)=\frac{1}{\hat{a}_{1}}\ln\left(1+\hat{a}_{1}S_{1}\right)+\frac{1}{\hat{a}_{2}}\ln\left(1+\hat{a}_{2}S_{2}\right)$$

$$S_{12}=\frac{1}{\hat{a}_{12}}\left[\left(1+\hat{a}_{1}S_{1}\right)^{\hat{a}_{12}/\hat{a}_{1}}\left(1+\hat{a}_{2}S_{2}\right)^{\hat{a}_{12}/\hat{a}_{2}}-1\right]$$

Tsallis - Nauenberg dispute

Nonextensive thermodynamics: a summary

$$\hat{L}(S_{12}) = \hat{L}(S_1) + \hat{L}(S_2) \qquad L(E_{12}) = L(E_1) + L(E_2)$$

$$\beta = \frac{1}{T} = \frac{\partial \hat{L}(S)}{\partial L(E)}$$

$$\hat{L}(S)[w_i] - \beta \sum_i w_i L(E_i) - \alpha \sum_i w_i = \max$$

Non-additive: Tsallis - Entropy

$$S_{\text{Tsallis}} = \frac{1}{\hat{a}} \sum_{i} \left(w_i^{1-\hat{a}} - w_i \right)$$

$$\hat{L}(S) = \frac{1}{\hat{a}} \ln \sum_{i} w_i^{1-\hat{a}} = S_{\text{Rényi}}$$

$$w_i^{eq} = \frac{1}{Z} \left(1 + \hat{a}(\beta E_i + \alpha) \right)^{-1/\hat{a}}$$

Power law factorizes → Energy is non-additive

$$w^{eq} = \frac{1}{\hat{Z}}\left(1 + \hat{a}\hat{\beta}E\right)^{-1/\hat{a}}$$

$$\left(1 + \hat{a}\hat{\beta}E_{12}\right)^{-1/\hat{a}} = \left(1 + \hat{a}\hat{\beta}E_{1}\right)^{-1/\hat{a}} \cdot \left(1 + \hat{a}\hat{\beta}E_{2}\right)^{-1/\hat{a}}$$

$$E_{12} = E_1 + E_2 + \hat{a}\hat{\beta}E_1E_2$$

Abstract Composition Rules

$$x \oplus y = h(x, y)$$

EPL 84: 56003, 2008

Repeated Composition, large-N

$$\text{Property:} \quad h(x,0) = x$$

$$\text{Repeated:} \quad x_N(y) = \underbrace{h \circ \ldots \circ h}_{N-1}(\Delta y_1, \ldots \Delta y_N)$$

$$y = \Delta y_1 + \ldots + \Delta y_N : \ \text{finite}$$

$$\text{Asymptotic rule:} \quad x_{N_1+N_2} = \varphi(x_{N_1}, x_{N_2})$$

Scaling law for large-N

$$x_n = h(x_{n-1}, \Delta y_n),\ x_0 = 0,\ \sum_{n=1}^{N} \Delta y_n = y$$

$$x_n - x_{n-1} = h(x_{n-1}, \Delta y_n) - h(x_{n-1}, 0)$$

$$x_n - x_{n-1} \approx \Delta y_n h_2'(x_{n-1}, 0^+)$$

$$N \to \infty : \boxed{\frac{dx}{dy} = h_2'(x, 0^+)}$$

Formal Logarithm

$$y = \int_0^x \frac{dz}{h_2'(z,0^+)} = L(x)$$

$$\varphi(x_1, x_2) = L^{-1}\left(L(x_1) + L(x_2)\right)$$

$$x_1 = x(y_1), \;\; x_2 = x(y_2),$$

$$\varphi = x(y_1 + y_2)$$

Asymptotic rules are associative and attractors among all rules…

Asymptotic rules are associative

$$\varphi(x,\varphi(y,z)) = \varphi(x, L^{-1}(L(y)+L(z)))$$
$$= L^{-1}\Big(L(x) + L\Big(L^{-1}(L(y)+L(z))\Big)\Big)$$
$$= L^{-1}(L(x)+L(y)+L(z))$$
$$= \ldots = \varphi(\varphi(x,y),z).$$

Associative rules are asymptotic

$$\Lambda(h(x,y)) = \Lambda(x) + \Lambda(y)$$

$$\Lambda'(h)\frac{\partial h}{\partial y} = \Lambda'(y)$$

$$h_2'(x,0) = \frac{\Lambda'(0)}{\Lambda'(h(x,0))} = \frac{\Lambda'(0)}{\Lambda'(x)}$$

$$L(x) = \int_0^x \frac{\Lambda'(z)}{\Lambda'(0)} dz = \frac{\Lambda(x)}{\Lambda'(0)}$$

$$\varphi(x,y) = h(x,y)$$

Scaled Formal Logarithm

$$L'(0) = 1, \quad L(0) = 0$$

$$L_a(x) = \frac{1}{a} L(ax)$$

$$L_a^{-1}(x) = \frac{1}{a} L^{-1}(ax)$$

$$L_0(x) = x$$

Deformed logarithm

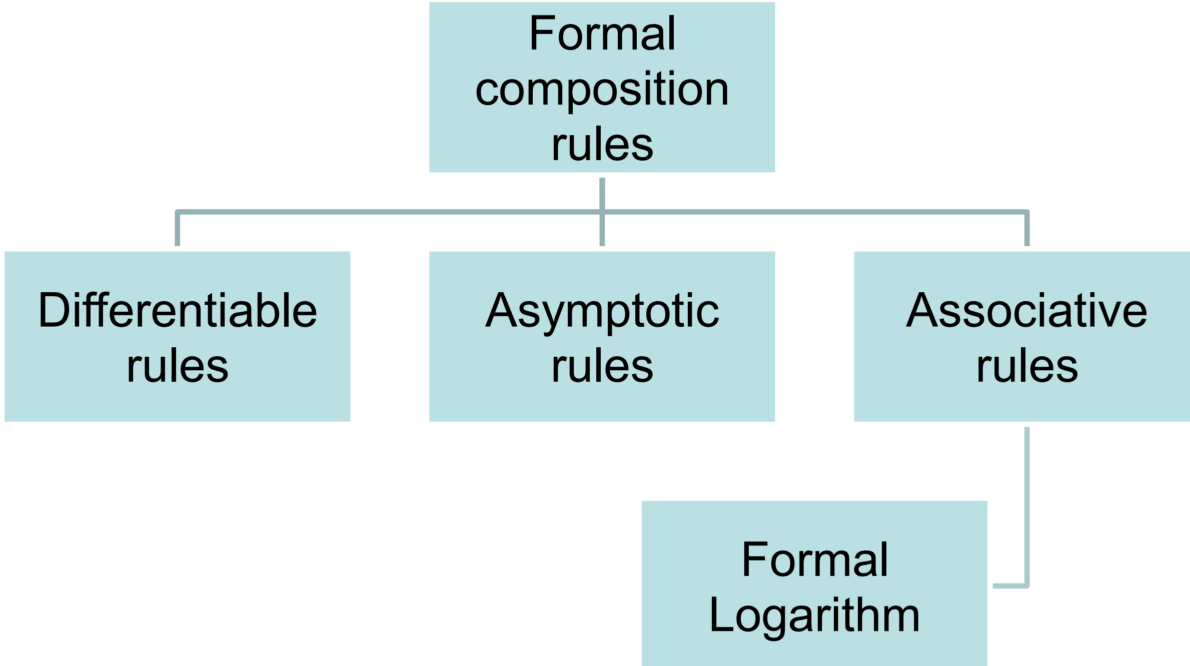
$$\ln_a(x) = L_a^{-1}(\ln(x))$$

$$\ln_a(1/x) = -\ln_{-a}(x)$$

Deformed exponential

$$e_a(x) = \exp(L_a(x))$$

$$1/e_a(x) = e_{-a}(-x)$$



1. **General rules repeated infinitely → asymptotic rules**
2. **Asymptotic rules are associative**
3. **Associative rules are self-asymptotic**
4. **For all associative rules there is a formal logarithm mapping it onto the simple addition**
5. **It can be obtained by scaling the general rule applied for small amounts**

Examples for composition rules

Example: Gibbs-Boltzmann

$$h(x, y) = x + y, \quad h_2'(x, 0^+) = 1$$

$$L(x) = x$$

$$f_{eq} = \frac{1}{Z} e(-\beta E)$$

$$S = -\int f \ln f$$

$$\text{for} \quad f = 1/W$$

$$S = k \ln W$$

Example: Rényi, Tsallis

$$h(x,y) = x + y + axy, \quad h_2'(x,0^+) = 1 + ax$$

$$L_a(x) = \frac{1}{a}\ln(1+ax), \quad f_{eq} = \frac{1}{Z}(1+aE)^{-\beta/a}$$

$$S_{non} = \frac{1}{a}\int (f^{1-a} - f)$$ Tsallis $q = 1 - a$

$$L_a(S_{non}) = \frac{1}{1-q}\ln\int f^q$$ Rényi

Example: Einstein

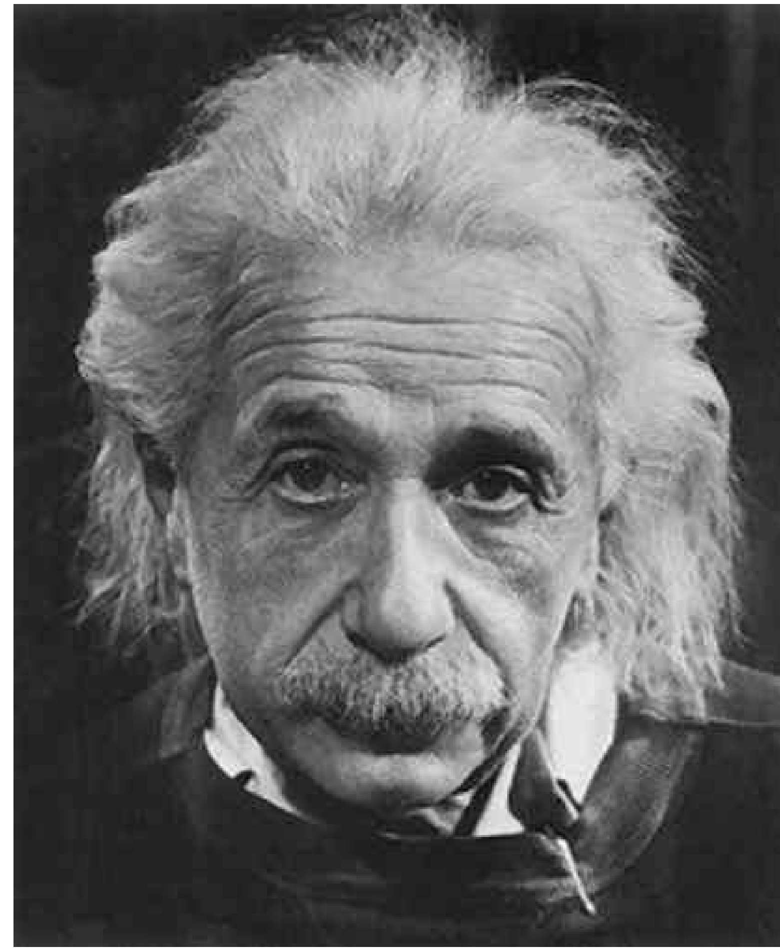
$$h(x,y) = \frac{x+y}{1+xy/c^2}$$

$$h_2'(x,0^+) = 1 - x^2/c^2$$

$$L_c(x) = c\,\mathrm{Ar}\tanh(\frac{x}{c})$$

$$L_c^{-1}(z) = c\tanh(\frac{z}{c})$$

$$\varphi(x,y) = h(x,y)$$



Example: Non associative

$$h(x,y) = x + y + a\frac{xy}{x+y}$$

$$h_2'(x,0^+) = 1 + a$$

$$L_c(x) = \frac{x}{1+a}$$

$$L_c^{-1}(z) = (1+a)z$$

$$\varphi(x,y) = x + y$$

Important example: product class

$$h(x, y) = x + y + G(xy)$$

$$h_2'(x, 0^+) = 1 + G'(0)\, x = 1 + ax$$

$$L_c(x) = \frac{1}{a} \ln(1 + ax)$$

$$L_c^{-1}(z) = \frac{e^{az} - 1}{a}$$

$$\varphi(x, y) = x + y + axy$$

Important example: product class

$$h(x,y) = x + y + G(xy)$$

$$h_2'(x,0^+) = 1 + G'(0)\,x = 1 + ax$$

$$L_c(x) = \frac{1}{a}\ln(1 + ax)$$

$$L_c^{-1}(z) = \frac{e^{az} - 1}{a}$$

$$\varphi(x,y) = x + y + axy$$

QCD is like this!

Relativistic energy composition

(high-energy limit: mass ≈ 0)

$$h(E_1, E_2) = E_1 + E_2 + U(Q^2)$$

$$Q^2 = (\vec{p}_1 - \vec{p}_2)^2 - (E_1 - E_2)^2$$

$$Q^2 = 2E_1E_2(1 - \cos\theta)$$

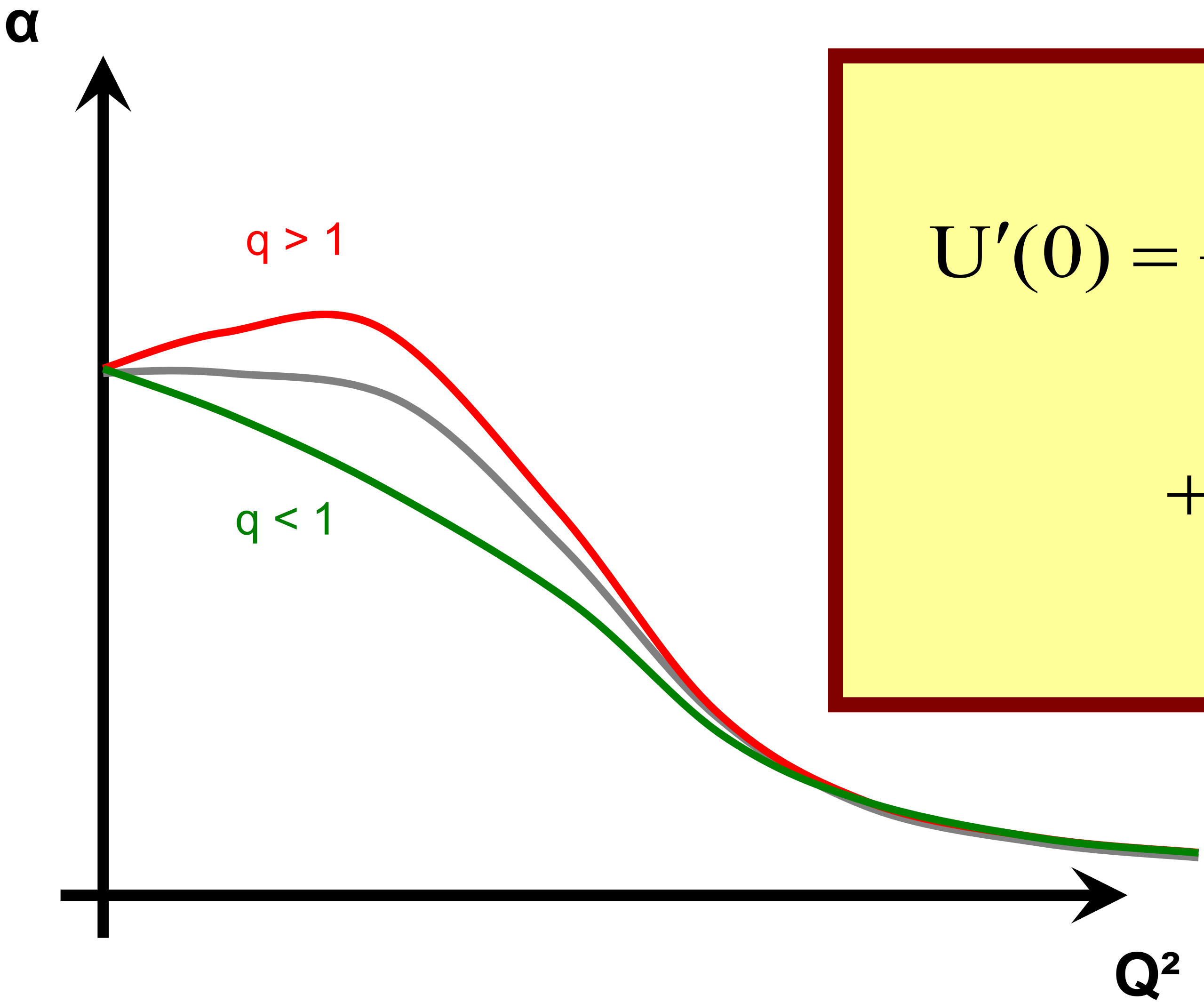
Asymptotic rule for m=0

$$h_2'(x,0^+) = 1 + 2xU'(0)$$

$$\varphi(x,y) = x + y + 2U'(0)xy$$

$$f_{eq} = \frac{1}{Z}\left(1 + 2U'(0)E\right)^{-\beta/2U'(0)}$$

Physics background:



$$U'(0) = \left.\frac{d\alpha}{dQ^2}\right|_{Q^2=0} \cdot \left\langle \frac{1}{r} \right\rangle + \left.\frac{d\kappa}{dQ^2}\right|_{Q^2=0} \cdot \langle r \rangle$$

Derivation as improved canonical

- **Derivation:**
  - Microcanonical entropy maximum for two
  - Reservoir-independent temperature: the best one can
  - Which composition rule leads to higher order agreement *(cannot be the simple addition)*
  - Make the choice of the additive L(S) universal → separation constant = 1 / heat capacity
  - Result: L(S) is Tsallis entropy, S is Rényi entropy

Derivation: formulas

- $S = -\sum_i P_i \ln P_i \quad \rightarrow \quad L(S) = \sum_i P_i \, \mathrm{L}(-\ln P_i)$
- $L\big(S(E_1)\big) + \; L\big(S(E - E_1)\big) \;\; = \;\; max.$
- $\beta_1 = L'(S(E_1)) \cdot S'(E_1)$

$$= L'\big(S(E \; - E_1)\big) \cdot S'(E - E_1)$$

Taylor: $S(E \; - E_1) = S(E) - E_1 S'(E) + \cdots$

Derivation: formulas

$$\beta_1 = L'(S(E)) \cdot S'(E) - E_1 \left[S'(E)^2 L''\big(S(E)\big) + S''(E) L'(S(E)) \right]$$

The content of the square bracket be **zero**!

Derivation: formulas

$$\beta = L'(S(E)) \cdot S'(E)$$

and the content of the bracket [ ] is **zero**:

$$\frac{L''(S)}{L'(S)} = -\frac{S''(E)}{S'(E)^2} = \frac{1}{C(E)}$$

***U*niversal *T*hermostat *I*ndependence:**

$$\boldsymbol{\frac{L''(S)}{L'(S)} = a}$$

Derivation: formulas

The solution is:

$$L(S) = \frac{e^{aS} - 1}{a}$$

This generates

$$\boldsymbol{L(-ln\, P_i) = \frac{1}{a}\,(P_i^{-a} - 1)}$$

Derivation: Tsallis entropy

The canonical principle becomes:

$$\frac{1}{a}\sum\left(P_i^{1-a} - P_i\right) - \beta \sum P_i E_i - \alpha \sum P_i = max.$$

The entropy with q = 1-a

$$S_{Tsallis} = \frac{1}{q-1}\sum\left(P_i - P_i^q\right)$$

Derivation: Rényi entropy

The Rényi entropy is the original one,

but the Tsallis entropy is to be maximized canonically

$$S_{Rényi} = L^{-1}(\, S_{Tsallis}\,) = \frac{1}{1-q}\, ln \sum P_i^q$$

Improved Canonical Distribution

- $P_i = \left(Z^{1-q} + (1-q)\frac{\beta}{q}\, E_i\right)^{\frac{1}{q-1}}$
- Expressed by the reservoir’s physical parameters via using our results:
- $P_i = \frac{1}{Z}\left(1 + \frac{Z^{-1/C}}{C-1}\, e^{S/C}\, \frac{1}{T}\, E_i\right)^{-C}$

Check infinite C limit!

Improved Logarithmic Slope

- $\frac{1}{\tau} = -\frac{d}{dE_i} \, ln \, P_i = \; T_0 \; + \; \frac{1}{C} \; E_i$

- Quark coalescence:

$$C_{meson} = 2 \, C_{quark}$$

$$C_{baryon} = 3 \, C_{quark}$$

- $T_0 = \; Te^{-S/C} \; Z^{1/C} \, (1 - {}^{1}/_{C})$

Check infinite C limit!

Infinite heat capacity limit

- $P_i \to \frac{1}{Z}\, e^{-E_i/T_{fit}}$ with
- $\boldsymbol{T_{fit} = \frac{1}{\beta} = T \lim_{C\to\infty} e^{-S/C}}$

Finite subsystem corrections to infinite heat capacity limit

- $$T_1 = T \frac{1}{1+1\cdot\frac{E_1}{CT}+\cdots}$$ traditional S-expansion

- $$T_1 = Te^{-S/C} \frac{e^{S(E_1)/C}}{1+0\cdot\frac{E_1}{CT}+\alpha\cdot\frac{E_1^2}{C^2T^2}+\cdots}$$ **Our expression**

Traditional: T1 < T, falling in E1; Ours: T1 < T, but rising in E1 !

Gaussian approximation

- Deviations from S=max equilibrium are traditionally considered as Gaussian:
- $\mathrm{P}(\Delta E) = \; e^{S(E_1)+S(E-E_1-\Delta E)} \;\approx$

$$e^{-S'(E-E_1)\,\Delta E+\frac{1}{2}S''(E-E_1)\,\Delta E^2} \;\approx$$

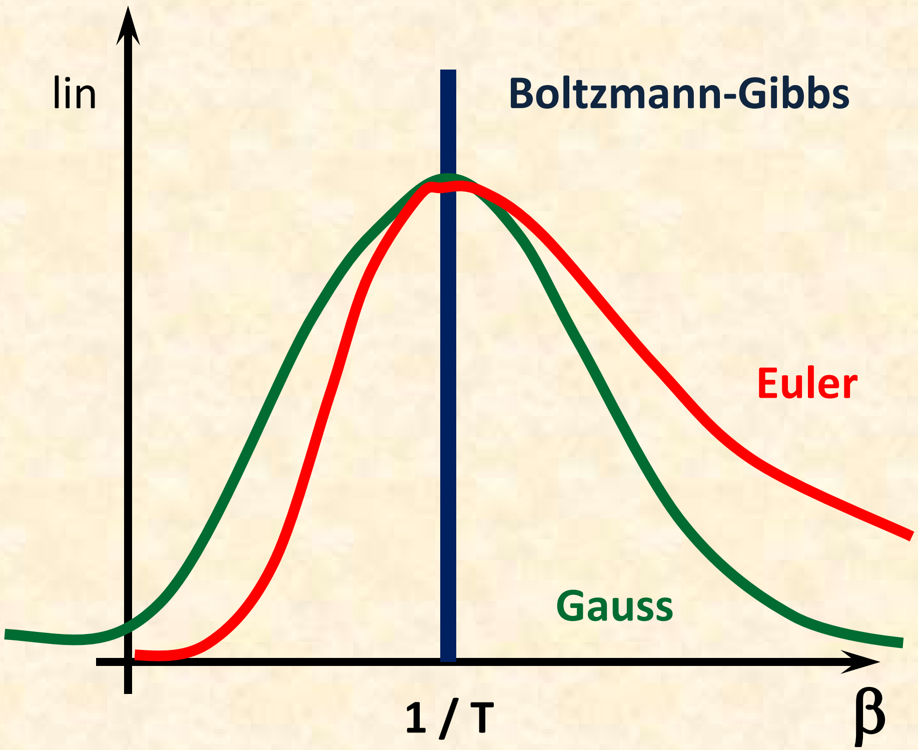
$$\propto \; e^{-\frac{1}{T}\Delta E-\frac{1}{2CT^2}\,\Delta E^2}$$

Gaussian approximation

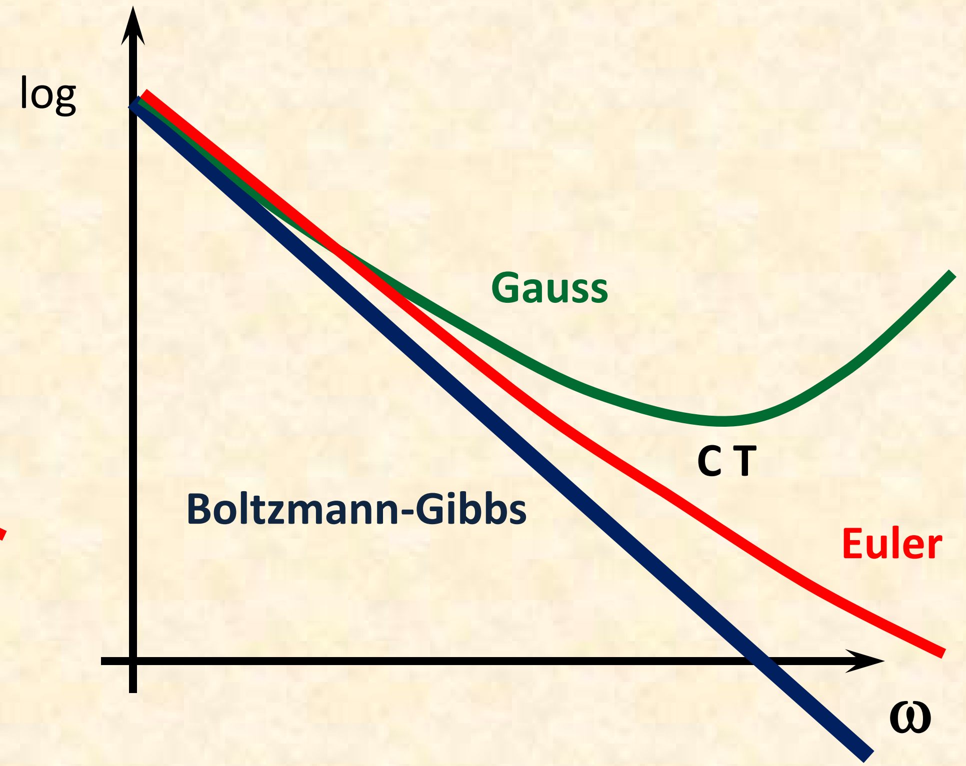
- After Legendre transformation also β fluctuates as Gaussian:
- $\mathrm{P}(\Delta\beta) \propto e^{-\frac{CT^2}{2}\Delta\beta^2+\cdots}$
- Thermodynamic "uncertainty" minimal

Gaussian approximation and beyond

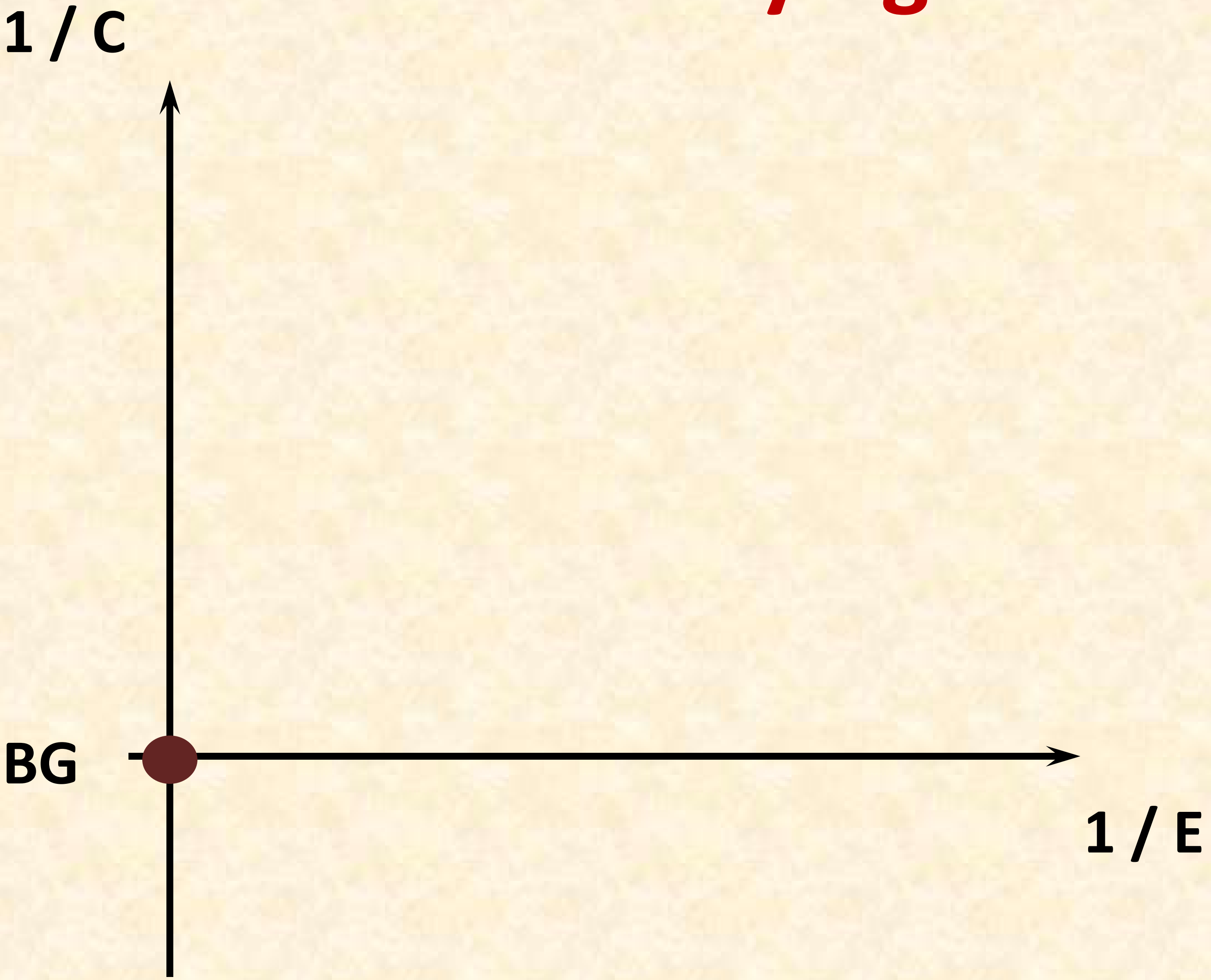
Beta fluctuation



Particle spectra : $\langle e^{-\beta\omega} \rangle$



Summary figure



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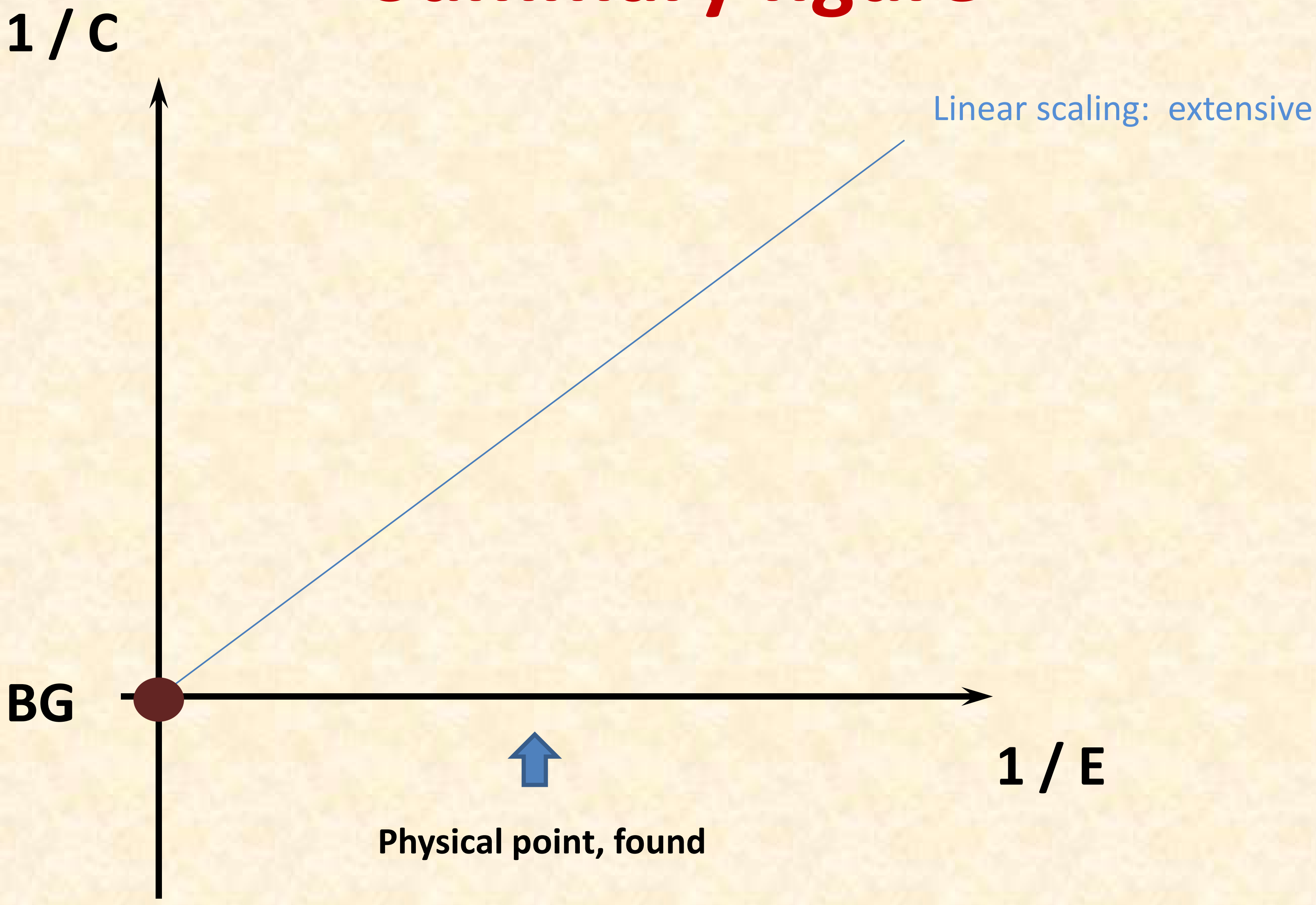

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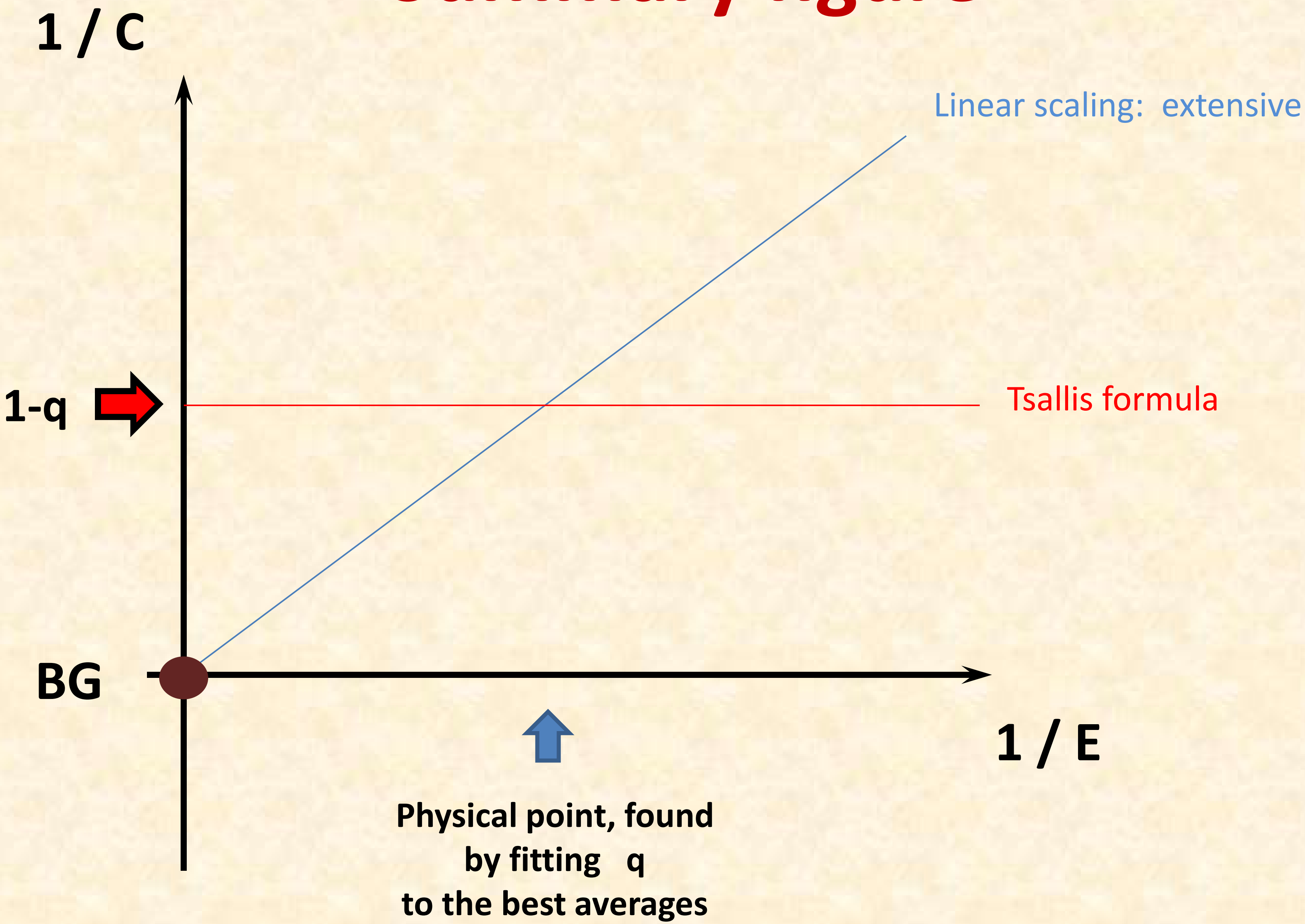

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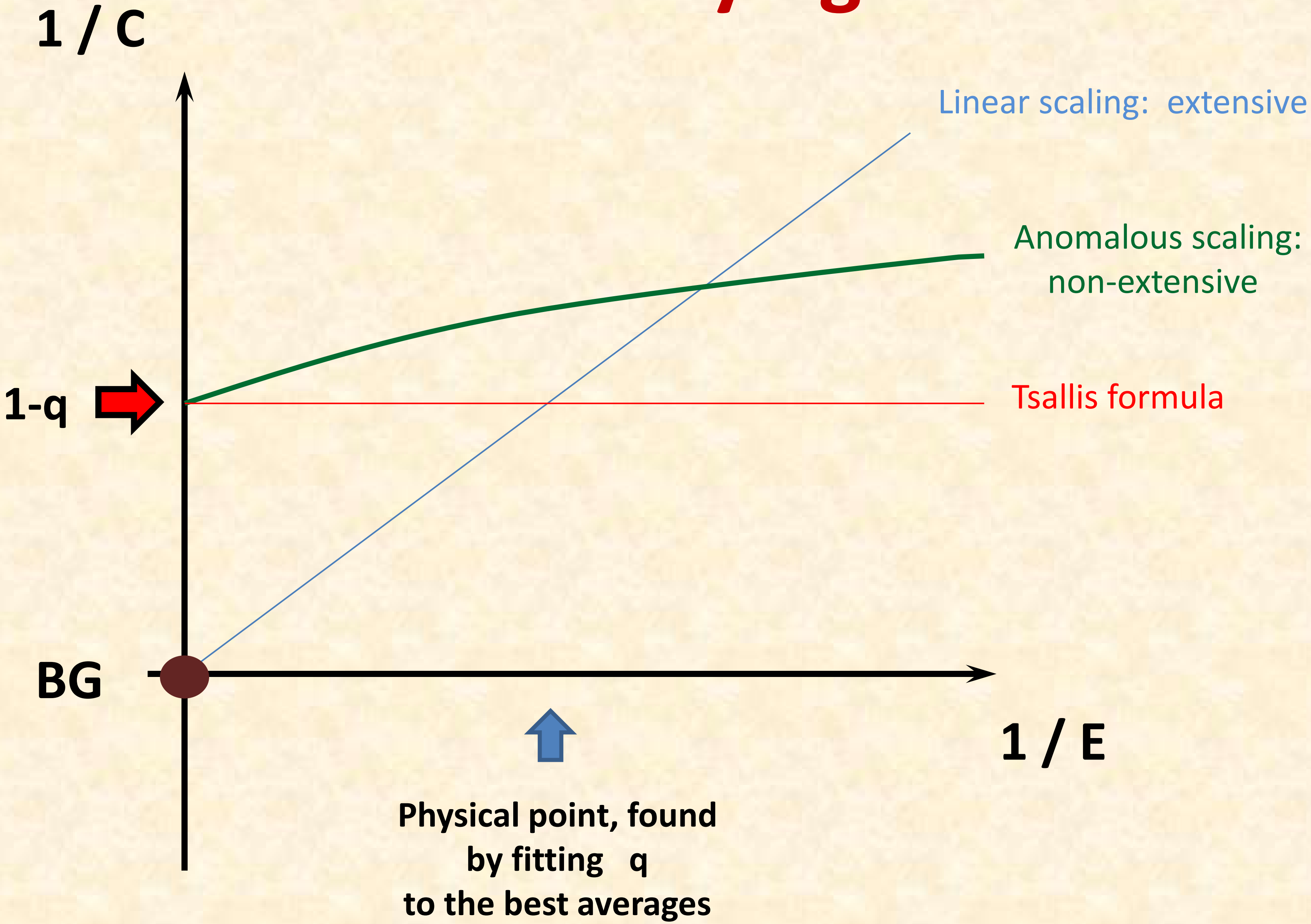

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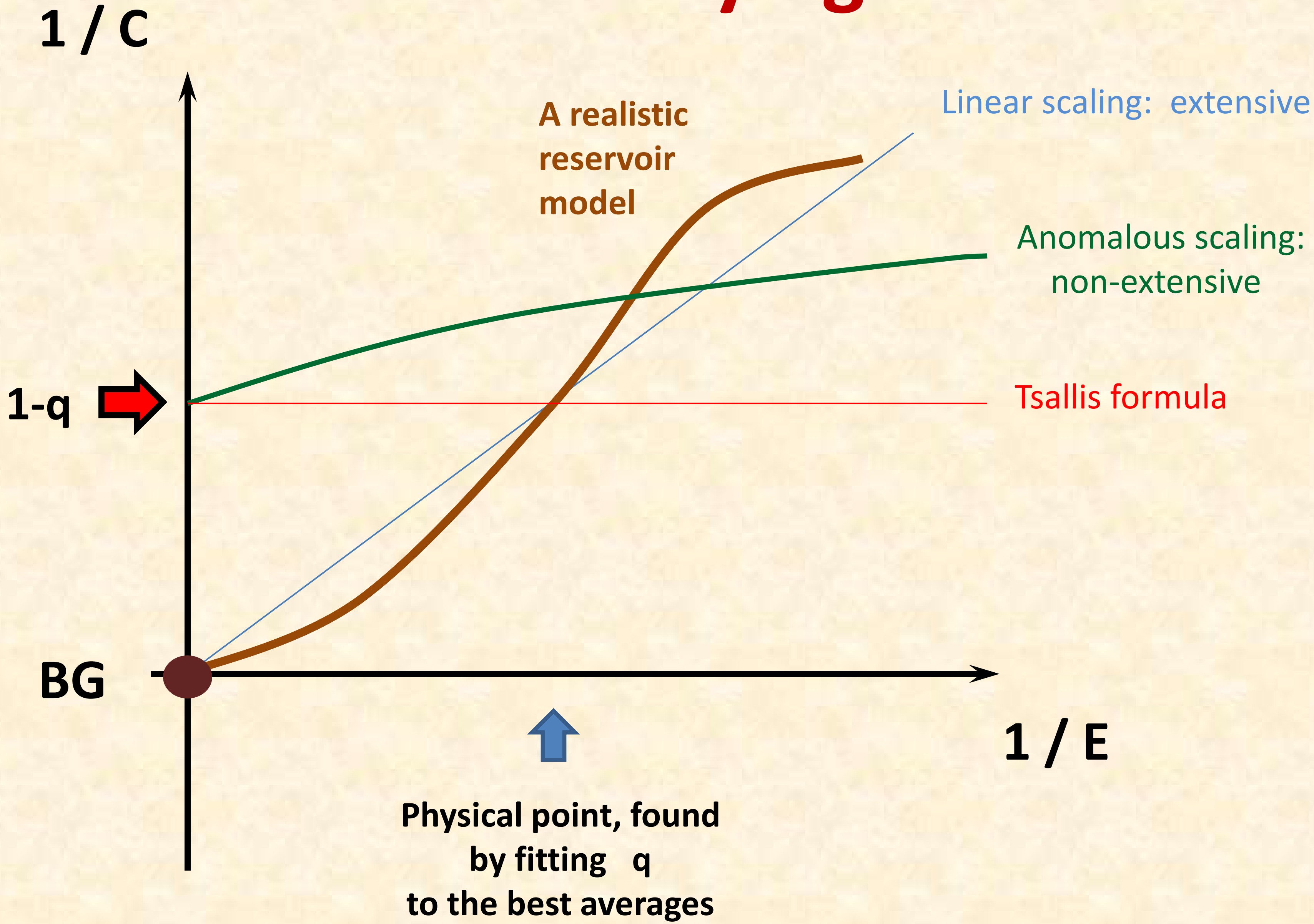

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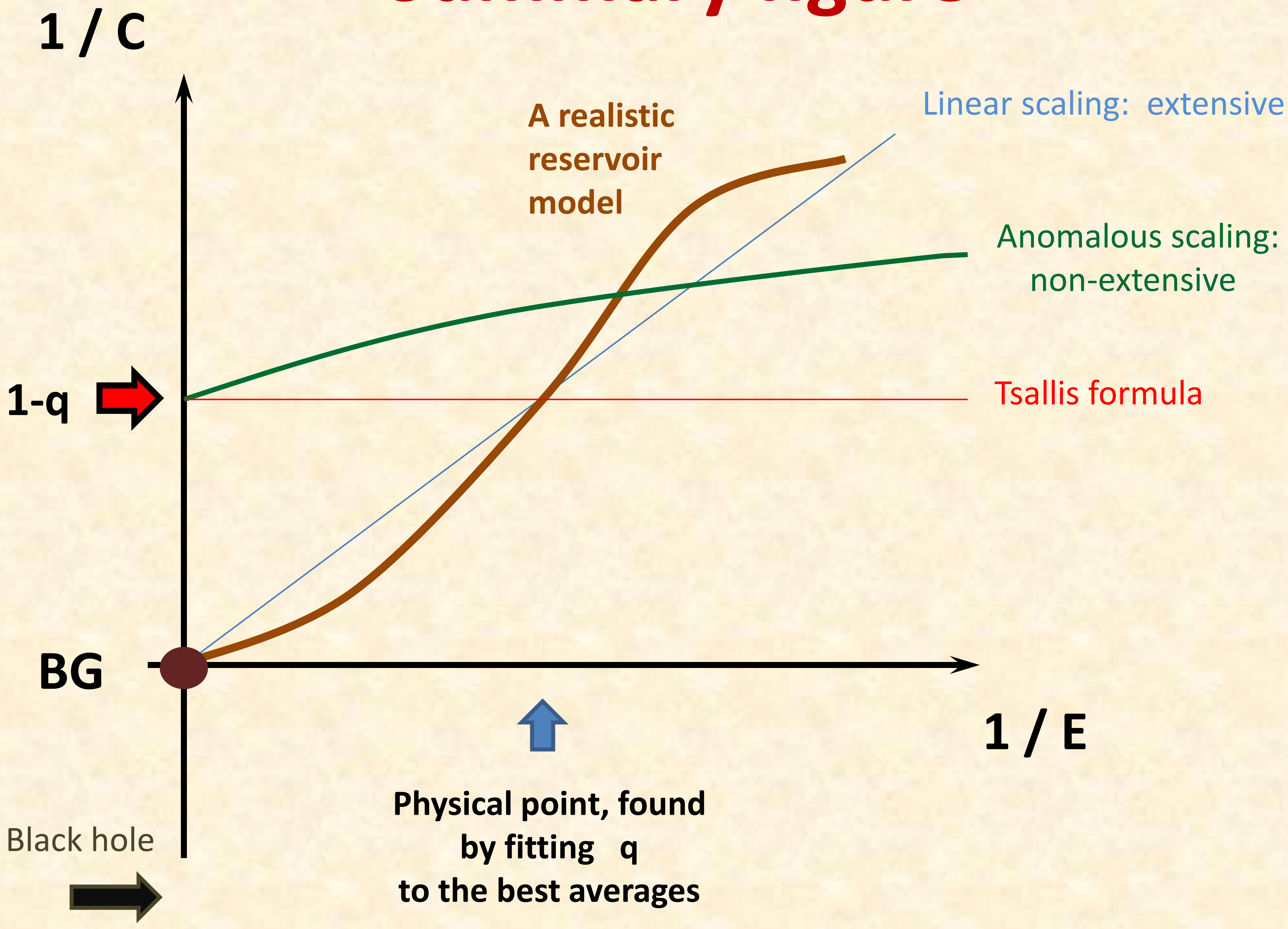
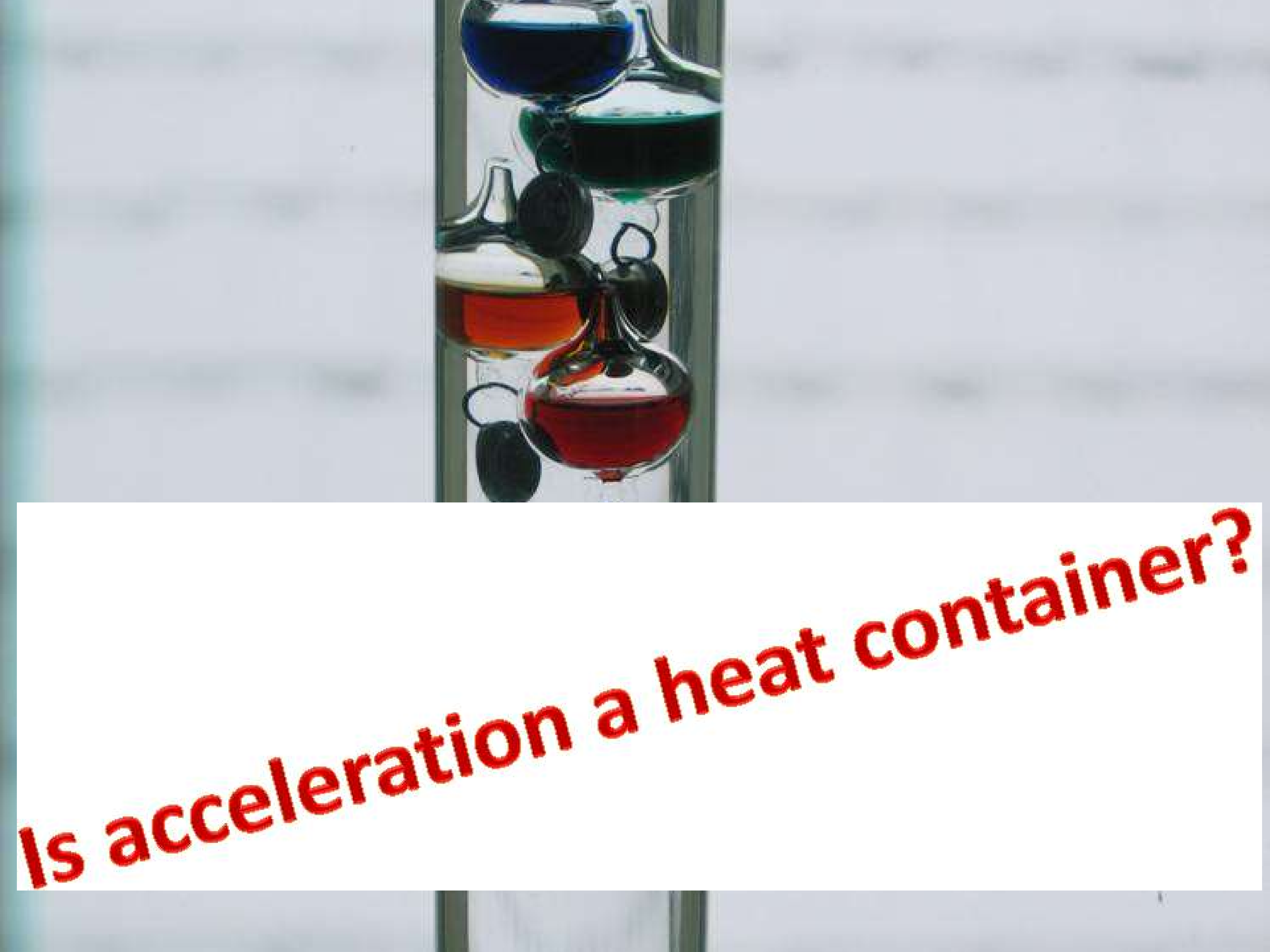



Discussion


Is acceleration a heat container?